

DIFFUSIVITY OF RESCALED RANDOM POLYMER IN RANDOM ENVIRONMENT IN DIMENSIONS 1 AND 2

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ABSTRACT. We show random polymer is diffusive in dimensions 1 and 2 in probability in an intermediate scaling regime. The scale is $\beta = o(N^{-1/4})$ in $d = 1$ and $\beta = o((\log N)^{-1/2})$ in $d = 2$ as $N \rightarrow \infty$.

1. INTRODUCTION

Consider walks $\omega : [0, N] \cap \mathbb{Z} \rightarrow \mathbb{Z}^d$ such that $\omega(0) = 0$, $|\omega(n) - \omega(n-1)| = 1$. Let P_0^N be uniform measure on the space of these walks each with weight $(2d)^{-N}$, then

$$p_0(N, x) := P_0^N(\omega(N) = x) = \int 1_{[\omega(N)=x]} dP_0^N(\omega) = \frac{1}{(2d)^N} \sum_{\omega: \omega(N)=x}$$

is probability of the nearest neighbor simple random walk starting at 0 is at site x at time N .

Let the random environment be given by $h = \{h(n, x) : n \in \mathbb{N}, x \in \mathbb{Z}^d\}$, a sequence of independent identically distributed random variables with $h(n, x) = \pm 1$ with equality probability on some probability space (H, \mathcal{G}, Q) , which are also independent of the simple random walk. We denote expectation over the environment space by E_Q .

We define the (unnormalized) polymer density by

$$p(N, x) = \int 1_{[\omega(N)=x]} \prod_{1 \leq n \leq N} [1 + c_{N,d} h(n, \omega(n))] dP_0^N(\omega)$$

where $c_{N,d}^1$ is such that

$$(1.1) \quad \lim_{N \rightarrow \infty} c_{N,1}^2 N^{1/2} = 0 \text{ for } d = 1; \quad \lim_{N \rightarrow \infty} c_{N,2}^2 \log N = 0 \text{ for } d = 2$$

Since the polymer density is not normalized, to obtain the probability of the polymer at time N is at site x , we define

$$p_N(N, x) = p(N, x)/Z(N)$$

where $Z(N)$ is the partition function

$$Z(N) = \sum_x p(N, x) = \int \prod_{1 \leq n \leq N} [1 + c_{N,d} h(n, \omega(n))] dP_0^N(\omega)$$

¹For example, we may take $c_{N,1} = N^{-(1/4+\epsilon)}$ and $c_{N,2} = \log N^{-(1/2+\epsilon)}$ for any $\epsilon > 0$. Also the scale $\beta = o(N^{-1/4})$ for $d = 1$ is first identified in [1]

In this paper, we show the mean square displacement of the polymer when scaled by N converges to 1 in probability in both $d = 1, 2$. Precisely, let $\langle \omega(N)^2 \rangle_{N,h} = \sum_x x^2 p_N(N, x)$,

Theorem 1.1. *With rescaling of the polymer density by $c_{N,d}$, for $d = 1, 2$,*

$$\frac{\langle \omega(N)^2 \rangle_{N,h}}{N} \rightarrow 1$$

in probability as $N \rightarrow \infty$.

We note that $\langle \omega(N)^2 \rangle_{N,h} = \frac{K(N)}{Z(N)}$, where $K(N) = \int \prod_{1 \leq n \leq N} [1 + \beta_{N,d} h(n, \omega(n))] \omega(N)^2 dP_0^N(\omega)$. To show the result, we are going to estimate second moment of the top and bottom quantity, and find that

Proposition 1.2. *For $d = 1$,*

$$i) E_Q(Z(N)^2) \leq \sum_{n=0}^N \left(c_1 c_{N,1}^2 N^{1/2} \right)^n; \quad ii) E_Q(K(N)^2) \leq N^2 \sum_{n=0}^N \left(c_1 c_{N,1}^2 N^{1/2} \right)^n$$

for some constant c_1 that depends only on the dimension.

Proposition 1.3. *For $d = 2$,*

$$i) E_Q(Z(N)^2) \leq \sum_{n=0}^N \left(c_2 c_{N,2}^2 \log N \right)^n; \quad ii) E_Q(K(N)^2) \leq N^2 \sum_{n=0}^N \left(c_2 c_{N,2}^2 \log N \right)^n$$

for some constants c_2 that depends only on the dimension.

The paper is organized as follows. In section 2, we write out second moments of the top and bottom quantity in the mean square displacement of the polymer. In Section 3, we show Proposition 1.2 and Theorem 1.1 for dimension $d = 1$. In Section 4, we show Proposition 1.3 and Theorem 1.1 for dimension $d = 2$. In Section 5, we show some other results.

2. SECOND MOMENT EXPANSIONS

In this section, we are going to write out the second moments of the top and bottom quantity in the mean square displacement of the polymer.

Lemma 2.1.

$$E_Q(Z^2(N)) = \sum_{n=0}^N \sum_{1 \leq i_1 < \dots < i_n \leq N} c_{N,d}^{2n} \sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0^2(i_k - i_{k-1}, x_k - x_{k-1})$$

Proof. By definition, $Z_N = \int \prod_{1 \leq n \leq N} [1 + c_{N,d} h(n, \omega(n))] dP_0^N(\omega)$. Upon expanding,

$$Z_N = \int \sum_{n=0}^N \sum_{1 \leq i_1 < \dots < i_n \leq N} c_{N,d}^n \prod_{k=1}^n h(i_k, \omega(i_k)) dP_0^N(\omega)$$

Let $f_n(\omega) = \sum_{1 \leq i_1 < \dots < i_n \leq N} c_{N,d}^n \prod_{k=1}^n h(i_k, \omega(i_k))$ and $g_n = \int f_n(\omega) dP_0^N(\omega)$, we see

$$Z_N^2 = (g_0 + g_1 + \dots + g_N)^2 = \sum_{0 \leq n, m \leq N} g_n g_m$$

For $n \neq m$, we have

$$E_Q g_n g_m = E_Q \int \sum_{1 \leq i_1 < \dots < i_n \leq N} c_{N,d}^n \prod_{k=1}^n h(i_k, \omega(i_k)) dP_0^N(\omega) \int \sum_{1 \leq i'_1 < \dots < i'_m \leq N} c_{N,d}^m \prod_{l=1}^m h(i'_l, \omega'(i'_l)) dP_0^N(\omega')$$

Note that if there is some i_k that is different from all other i'_l 's (or vice versa), then by independence of the $h(n, x)$'s and that they have mean 0, we have

$$E_Q \prod_{k=1}^n \prod_{l=1}^m h(i_k, \omega(i_k)) h(i'_l, \omega'(i'_l)) = 0$$

By Fubini, $E_Q g_n g_m = 0$. But since $n \neq m$, the i_k 's and i'_l 's cannot all be matched in pairs, so there must be some i_k different from all other i'_l 's (or vice versa).

On the other hand, for $n = m$, we have

$$\begin{aligned} E_Q g_n^2 &= E_Q \int \sum_{1 \leq i_1 < \dots < i_n \leq N} c_{N,d}^n \prod_{k=1}^n h(i_k, \omega(i_k)) dP_0^N(\omega) \int \sum_{1 \leq i'_1 < \dots < i'_n \leq N} c_{N,d}^n \prod_{k=1}^n h(i'_k, \omega'(i'_k)) dP_0^N(\omega') \\ &= E_Q \int \int \sum_{1 \leq i_1 < \dots < i_n \leq N} c_{N,d}^{2n} \prod_{k=1}^n h(i_k, \omega(i_k)) h(i_k, \omega'(i_k)) dP_0^N(\omega) dP_0^N(\omega') \\ &\quad + E_Q \int \int \sum_{i_l \neq i'_l \text{ for some } l \in \{1, \dots, n\}} c_{N,d}^{2n} \prod_{k=1}^n h(i_k, \omega(i_k)) h(i'_k, \omega'(i'_k)) dP_0^N(\omega) dP_0^N(\omega') \\ &= E_Q \int \int \sum_{1 \leq i_1 < \dots < i_n \leq N} c_{N,d}^{2n} \prod_{k=1}^n 1_{\omega(i_k) = \omega'(i_k)} dP_0^N(\omega) dP_0^N(\omega') \\ &\quad + E_Q \int \int \sum_{i_l \neq i'_l \text{ for some } l \in \{1, \dots, n\}} c_{N,d}^{2n} \prod_{k=1}^n h(i_k, \omega(i_k)) h(i'_k, \omega'(i'_k)) dP_0^N(\omega) dP_0^N(\omega') \\ &= \int \int \sum_{1 \leq i_1 < \dots < i_n \leq N} c_{N,d}^{2n} \prod_{k=1}^n 1_{\omega(i_k) = \omega'(i_k)} dP_0^N(\omega) dP_0^N(\omega') \end{aligned}$$

Third equality follows because $h^2(n, x) = 1$ and nonzero contribution only comes from when all sites $\omega(i_k)$ and $\omega'(i_k)$ are matched in pairs. Fourth equality follows because if the i_α 's were to match perfectly with the i'_β 's for $\alpha \neq \beta$, then we would get contradiction in terms of the order of the times. For example, take $n = 3$ and the perfect cross matching $i_1 = i'_2$, $i_2 = i'_3$, $i_3 = i'_1$, then by $i_1 < i_2 < i_3$ we would have $i'_2 < i'_3 < i'_1$, which is a contradiction.

Now, we are going to write out the integrals as sums in terms of the transition probabilities of the two independent walks. By above, we have

$$\begin{aligned} E_Q(Z^2(N)) &= \sum_{n=0}^N \sum_{1 \leq i_1 < \dots < i_n \leq N} c_{N,d}^{2n} \int \int 1_{[\omega(i_1) = \tilde{\omega}(i_1), \dots, \omega(i_n) = \tilde{\omega}(i_n)]} dP_0^N(\omega) dP_0^N(\tilde{\omega}) \\ &= \sum_{n=0}^N \sum_{1 \leq i_1 < \dots < i_n \leq N} c_{N,d}^{2n} \int \sum_{x_1, \dots, x_n, x} 1_{[\omega(i_1) = x_1, \dots, \omega(i_n) = x_n]} \\ &\quad \times P_0^N(\tilde{\omega}(i_1) = x_1, \dots, \tilde{\omega}(i_n) = x_n, \tilde{\omega}(N) = x) dP_0^N(\omega) \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^N \sum_{1 \leq i_1 < \dots < i_n \leq N} c_{N,d}^{2n} \int \sum_{x_1, \dots, x_n} 1_{[\omega(i_1)=x_1, \dots, \omega(i_n)=x_n]} \\
&\times \prod_{k=1}^n p_0(i_k - i_{k-1}, x_k - x_{k-1}) \sum_x p_0(N - i_n, x - x_n) dP_0^N(\omega)
\end{aligned}$$

where in the first equality we also need to sum over sites at time N because P_0^N is measure for walks of length N , and in the last equality we use the fact that increments of the walk are independent, and the walk is spatial homogeneous, i.e. probability of the walk starting at y and ending at x is same as probability of the walk starting at 0 and ending at $y - x$. Next we note that $\sum_x p_0(N - i_n, x - x_n) = 1$ because $p_0(n, x)$ is a transition probability.

Combining above and expanding similarly for the second walk we thus have shown Lemma 2.1. \square

Lemma 2.2.

$$\begin{aligned}
E_Q(K^2(N)) &= \sum_{n=0}^N \sum_{1 \leq i_1 < \dots < i_n \leq N} c_{N,d}^{2n} \sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0^2(i_k - i_{k-1}, x_k - x_{k-1}) \\
&\times \left(\sum_x x^2 p_0(N - i_n, x - x_n) \right)^2
\end{aligned}$$

Proof. To estimate second moment of $K(N)$, we have

$$\begin{aligned}
E_Q(K^2(N)) &= E_Q \int \int \prod_{1 \leq n \leq N} [1 + c_{N,d} h(n, \omega(n))] [1 + c_{N,d} h(n, \tilde{\omega}(n))] \\
&\times \omega(N)^2 \tilde{\omega}(N)^2 dP_0^N(\omega) dP_0^N(\tilde{\omega})
\end{aligned}$$

As we see, the only difference between $E_Q(Z^2(N))$ and $E_Q(K^2(N))$ is the extra term $\omega(N)^2 \tilde{\omega}(N)^2$, and we proceed as before to expand the second moment to get

$$\begin{aligned}
&\sum_{n=0}^N \sum_{1 \leq i_1 < \dots < i_n \leq N} c_{N,d}^{2n} \int \int 1_{[\omega(i_1)=\tilde{\omega}(i_1), \dots, \omega(i_n)=\tilde{\omega}(i_n)]} \omega(N)^2 \tilde{\omega}(N)^2 dP_0^N(\omega) dP_0^N(\tilde{\omega}) \\
&= \sum_{n=0}^N \sum_{1 \leq i_1 < \dots < i_n \leq N} c_{N,d}^{2n} \int \sum_{x_1, \dots, x_n, x} 1_{[\omega(i_1)=x_1, \dots, \omega(i_n)=x_n]} \\
&\times x^2 P_0^N(\tilde{\omega}(i_1) = x_1, \dots, \tilde{\omega}(i_n) = x_n, \tilde{\omega}(N) = x) \omega(N)^2 dP_0^N(\omega) \\
&= \sum_{n=0}^N \sum_{1 \leq i_1 < \dots < i_n \leq N} c_{N,d}^{2n} \int \sum_{x_1, \dots, x_n} 1_{[\omega(i_1)=x_1, \dots, \omega(i_n)=x_n]} \\
&\times \prod_{k=1}^n p_0(i_k - i_{k-1}, x_k - x_{k-1}) \sum_x x^2 p_0(N - i_n, x - x_n) \omega(N)^2 dP_0^N(\omega) \\
&= \sum_{n=0}^N \sum_{1 \leq i_1 < \dots < i_n \leq N} c_{N,d}^{2n} \sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0^2(i_k - i_{k-1}, x_k - x_{k-1}) \\
&\times \sum_x x^2 p_0(N - i_n, x - x_n) \sum_y y^2 p_0(N - i_n, y - x_n)
\end{aligned}$$

□

3. DIFFUSIVITY OF RESCALED RANDOM POLYMER IN $d = 1$

In this section, we are going to show Proposition 1.2 and Theorem 1.1 for dimension $d = 1$.

The key ingredient we need is that the transition probability $p_0(n, x)$ has the following estimate by the Gaussian density, more precisely, for $d \geq 1$, $x \in \mathbb{Z}^d$ such that $x_1 + \dots + x_d + n \equiv 0 \pmod{2}$, then

$$(3.2) \quad p_0(n, x) = 2 \left(\frac{d}{2\pi n} \right)^{d/2} \exp \left(-\frac{d|x|^2}{2n} \right) + r_n(x)$$

where $|r_n(x)| \leq \min(c_d n^{-(d+2)/2}, c'_d |x|^{-2} n^{-d/2})$ for some constants c_d, c'_d that depend only on the dimension. (See Theorem 1.2.1 in [2])

Section 3.1. In this subsection, we are going to show Proposition 1.2 i) in a series of lemmas.

Lemma 3.1.

$$\sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0^2(i_k - i_{k-1}, x_k - x_{k-1}) \leq c_1^n \prod_{k=1}^n (i_k - i_{k-1})^{-1/2}$$

for some constant c_1 that depends only on the dimension $d = 1$

Proof. For $d = 1$, since $e^{-x^2} \leq 1$ for all x , we see (3.2) is at most $c_1 n^{-1/2}$ for some constant c_1 . Using this uniform estimate for each of the $p_0(i_k - i_{k-1}, x_k - x_{k-1})$'s, we have

$$\begin{aligned} \sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0^2(i_k - i_{k-1}, x_k - x_{k-1}) &= \sum_{x_1} p_0^2(i_1, x_1) \cdots \sum_{x_n} p_0^2(i_n - i_{n-1}, x_n - x_{n-1}) \\ &\leq c_1^n i_1^{-1/2} \cdots (i_n - i_{n-1})^{-1/2} \sum_{x_1} p_0(i_1, x_1) \cdots \sum_{x_n} p_0(i_n - i_{n-1}, x_n - x_{n-1}) = c_1^n \prod_{k=1}^n (i_k - i_{k-1})^{-1/2} \end{aligned}$$

for some constant c_1 that depends only on the dimension $d = 1$ (last equality follows from that $p_0(n, x)$ is a transition probability). □

Lemma 3.2.

$$\sum_{1 \leq i_1 < \dots < i_n \leq N} c_1^n c_{N,1}^{2n} \prod_{k=1}^n (i_k - i_{k-1})^{-1/2} \leq \left(c_1 c_{N,1}^2 N^{1/2} \right)^n$$

Proof.

$$\begin{aligned} &\sum_{1 \leq i_1 < \dots < i_n \leq N} c_1^n c_{N,1}^{2n} \prod_{k=1}^n (i_k - i_{k-1})^{-1/2} \\ &= c_1^n c_{N,1}^{2n} \sum_{i_1=1}^{N-(n-1)} \cdots \sum_{i_{n-1}=i_{n-2}+1}^{N-1} i_1^{-1/2} \cdots (i_{n-1} - i_{n-2})^{-1/2} \sum_{i_n=i_{n-1}+1}^N (i_n - i_{n-1})^{-1/2} \end{aligned}$$

$$\leq c_1^n c_{N,1}^{2n} \sum_{i_1=1}^{N-(n-1)} \cdots \sum_{i_{n-1}=i_{n-2}+1}^{N-1} i_1^{-1/2} \cdots (i_{n-1} - i_{n-2})^{-1/2} 2N^{1/2}$$

Last inequality holds because $\sum_{k=1}^N k^{-1/2} \leq 1 + \int_1^N x^{-1/2} dx = 1 + 2(N^{1/2} - 1) \leq 2N^{1/2}$. Continuing from above and arguing similarly to estimate each sum in the expression we have

$$\leq c_1^n c_{N,1}^{2n} N^{1/2} \sum_{i_1=1}^{N-(n-1)} \cdots \sum_{i_{n-1}=i_{n-2}+1}^{N-1} i_1^{-1/2} \cdots (i_{n-1} - i_{n-2})^{-1/2} \leq \left(c_1 c_{N,1}^2 N^{1/2}\right)^n$$

where the constant c_1 will change from line to line (again it depends only on the dimension $d = 1$). \square

We conclude by Lemma 2.1 that Proposition 1.2 i) holds.

Section 3.2. In this subsection, we are going to show Proposition 1.2 ii) in a series of lemmas.

By standard computations of the moments of simple random walk of length n in dimension $d = 1$ using characteristic function, we have

$$\sum_x x^2 p_0(n, x) = n; \quad \sum_x x^4 p_0(n, x) = 3n^2 - 2n$$

Lemma 3.3.

$$\sum_x x^2 p_0(N - i_n, x - x_n) = (N - i_n) + x_n^2$$

Proof.

$$\begin{aligned} \sum_x x^2 p_0(N - i_n, x - x_n) &= \sum_{x-x_n} x^2 p_0(N - i_n, x - x_n) = \sum_x (x + x_n)^2 p_0(N - i_n, x) \\ &= \sum_x x^2 p_0(N - i_n, x) + \sum_x x_n^2 p_0(N - i_n, x) = (N - i_n) + x_n^2 \end{aligned}$$

where first equality holds because summation over all x 's is same as summation over all $x - x_n$'s and third equality because holds any odd moment of simple random walk vanish. \square

Lemma 3.4.

$$\begin{aligned} &\sum_{x_k} x_k^4 p_0(i_k - i_{k-1}, x_k - x_{k-1}) \\ &= 3(i_k - i_{k-1})^2 - 2(i_k - i_{k-1}) + 6x_{k-1}^2(i_k - i_{k-1}) + x_{k-1}^4 \end{aligned}$$

Proof.

$$\begin{aligned} \sum_{x_k} x_k^4 p_0(i_k - i_{k-1}, x_k - x_{k-1}) &= \sum_{x_k - x_{k-1}} x_k^4 p_0(i_k - i_{k-1}, x_k - x_{k-1}) = \sum_{x_k} (x_k + x_{k-1})^4 p_0(i_k - i_{k-1}, x_k) \\ &= \sum_{x_k} x_k^4 p_0(i_k - i_{k-1}, x_k) + \sum_{x_k} 6x_k^2 x_{k-1}^2 p_0(i_k - i_{k-1}, x_k) + \sum_{x_k} x_{k-1}^4 p_0(i_k - i_{k-1}, x_k) \\ &= 3(i_k - i_{k-1})^2 - 2(i_k - i_{k-1}) + 6x_{k-1}^2(i_k - i_{k-1}) + x_{k-1}^4 \end{aligned}$$

□

Lemma 3.5.

$$\begin{aligned}
& \sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0(i_k - i_{k-1}, x_k - x_{k-1}) ((N - i_n)^2 + 2(N - i_n)x_n^2 + x_n^4) \\
&= (N - i_n)^2 + 2(N - i_n)i_n + 3 \sum_{k=1}^n (i_k - i_{k-1})^2 - 2i_n + 6 \sum_{k=1}^n (i_k - i_{k-1})i_{k-1}
\end{aligned}$$

Proof. We do this by induction on n . For $n = 1$, we have

$$\begin{aligned}
& \sum_{x_1} p_0(i_1, x_1) ((N - i_1)^2 + 2(N - i_1)x_1^2 + x_1^4) \\
&= (N - i_1)^2 \sum_{x_1} p_0(i_1, x_1) + 2(N - i_1) \sum_{x_1} x_1^2 p_0(i_1, x_1) + \sum_{x_1} x_1^4 p_0(i_1, x_1) \\
&= (N - i_1)^2 + 2(N - i_1)i_1 + 3i_1^2 - 2i_1
\end{aligned}$$

(Note we do not have term of the form $6 \sum_{k=1}^n (i_k - i_{k-1})i_{k-1}$ because $i_0 = 0$.)

Suppose equality holds for $n - 1$. Then

$$\begin{aligned}
& \sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0(i_k - i_{k-1}, x_k - x_{k-1}) ((N - i_n)^2 + 2(N - i_n)x_n^2 + x_n^4) \\
&= \sum_{x_1, \dots, x_{n-1}} \prod_{k=1}^{n-1} p_0(i_k - i_{k-1}, x_k - x_{k-1}) \sum_{x_n} p_0(i_n - i_{n-1}, x_n - x_{n-1}) ((N - i_n)^2 + 2(N - i_n)x_n^2 + x_n^4) \\
&= \sum_{x_1, \dots, x_{n-1}} \prod_{k=1}^{n-1} p_0(i_k - i_{k-1}, x_k - x_{k-1}) [(N - i_n)^2 + 2(N - i_n)(i_n - i_{n-1} + x_{n-1}^2) \\
&\quad + 3(i_n - i_{n-1})^2 - 2(i_n - i_{n-1}) + 6x_{n-1}^2(i_n - i_{n-1}) + x_{n-1}^4] \\
&= \sum_{x_1, \dots, x_{n-1}} \prod_{k=1}^{n-1} p_0(i_k - i_{k-1}, x_k - x_{k-1}) [(N - i_n)^2 + 2(N - i_n)(i_n - i_{n-1}) + 3(i_n - i_{n-1})^2 - 2(i_n - i_{n-1})] \\
&\quad + [2(N - i_n) + 6(i_n - i_{n-1})]x_{n-1}^2 + x_{n-1}^4 \\
&= [(N - i_n)^2 + 2(N - i_n)(i_n - i_{n-1}) + 3(i_n - i_{n-1})^2 - 2(i_n - i_{n-1})] + [2(N - i_n) + 6(i_n - i_{n-1})]i_{n-1} \\
&\quad + 3 \sum_{k=1}^{n-1} (i_k - i_{k-1})^2 - 2i_{n-1} + 6 \sum_{k=1}^{n-1} (i_k - i_{k-1})i_{k-1} \\
&= (N - i_n)^2 + 2(N - i_n)(i_n - i_{n-1} + i_{n-1}) + 3(i_n - i_{n-1})^2 + 3 \sum_{k=1}^{n-1} (i_k - i_{k-1})^2 \\
&\quad - 2(i_n - i_{n-1} + i_{n-1}) + 6(i_n - i_{n-1})i_{n-1} + 6 \sum_{k=1}^{n-1} (i_k - i_{k-1})i_{k-1} \\
&= (N - i_n)^2 + 2(N - i_n)i_n + 3 \sum_{k=1}^n (i_k - i_{k-1})^2 - 2i_n + 6 \sum_{k=1}^n (i_k - i_{k-1})i_{k-1}
\end{aligned}$$

where in the second equality we use Lemmas 3.3 and 3.4 and in the fourth equality we use the inductive hypothesis. □

Lemma 3.6.

$$(N - i_n)^2 + 2(N - i_n)i_n + 3 \sum_{k=1}^n (i_k - i_{k-1})^2 - 2i_n + 6 \sum_{k=1}^n (i_k - i_{k-1})i_{k-1} \leq 100^n N^2$$

Proof.

$$\begin{aligned} & (N - i_n)^2 + 2(N - i_n)i_n + 3 \sum_{k=1}^n (i_k - i_{k-1})^2 - 2i_n + 6 \sum_{k=1}^n (i_k - i_{k-1})i_{k-1} \\ &= N^2 - 2Ni_n + i_n^2 + 2Ni_n - 2i_n^2 + 3 \sum_{k=1}^n (i_k^2 - 2i_k i_{k-1} + i_{k-1}^2) - 2i_n + 6 \sum_{k=1}^n (i_k i_{k-1} - i_{k-1}^2) \\ &\leq N^2 + 2N^2 + N^2 + 2N^2 + 2N^2 + 3n(N^2 + 2N^2 + N^2) + 2N^2 + 6n(N^2 + N^2) \\ &= 10N^2 + 24nN^2 \leq 100^n N^2 \end{aligned}$$

□

Lemma 3.7.

$$\sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0^2(i_k - i_{k-1}, x_k - x_{k-1}) \left(\sum_x x^2 p_0(N - i_n, x - x_n) \right)^2 \leq c_1^n N^2 \prod_{k=1}^n (i_k - i_{k-1})^{-1/2}$$

Proof. Using the uniform estimate (3.2) for $d = 1$ on the transition probability as in the proof of Lemma 3.1, we get

$$\begin{aligned} & \sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0^2(i_k - i_{k-1}, x_k - x_{k-1}) \left(\sum_x x^2 p_0(N - i_n, x - x_n) \right)^2 \\ &\leq c_1^n i_1^{-1/2} (i_2 - i_1)^{-1/2} \dots (i_n - i_{n-1})^{-1/2} \sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0(i_k - i_{k-1}, x_k - x_{k-1}) ((N - i_n) + x_n^2)^2 \\ &= c_1^n i_1^{-1/2} (i_2 - i_1)^{-1/2} \dots (i_n - i_{n-1})^{-1/2} \sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0(i_k - i_{k-1}, x_k - x_{k-1}) \\ &\quad \times ((N - i_n)^2 + 2(N - i_n)x_n^2 + x_n^4) \\ &= c_1^n i_1^{-1/2} (i_2 - i_1)^{-1/2} \dots (i_n - i_{n-1})^{-1/2} \\ &\quad \times \left((N - i_n)^2 + 2(N - i_n)i_n + 3 \sum_{k=1}^n (i_k - i_{k-1})^2 - 2i_n + 6 \sum_{k=1}^n (i_k - i_{k-1})i_{k-1} \right) \\ &\leq c_1^n N^2 \prod_{k=1}^n (i_k - i_{k-1})^{-1/2} \end{aligned}$$

where in the first inequality we also use Lemma 3.3 to compute the second moment, in the second equality we use Lemma 3.5 and in the last inequality we use Lemma 3.6. □

Lemma 3.8.

$$\sum_{1 \leq i_1 < \dots < i_n \leq N} c_1^n c_{N,1}^{2n} N^2 \prod_{k=1}^n (i_k - i_{k-1})^{-1/2} \leq N^2 \left(c_1 c_{N,1}^2 N^{1/2} \right)^n$$

Proof. It follows from Lemma 3.2. □

We conclude by Lemma 2.2 that Proposition 1.2 ii) holds.

Section 3.3. In this subsection, we are going to use Proposition 1.2 to show Theorem 1.1 for $d = 1$. We do so with a series of lemmas.

Lemma 3.9.

$$E_Q((Z(N) - 1)^2) \leq \sum_{n=1}^N \left(c_1 c_{N,1}^2 N^{1/2} \right)^n$$

Proof.

$$\begin{aligned} E_Q((Z(N) - 1)^2) &= E_Q(Z^2(N) - 2Z(N) + 1) \\ &= E_Q(Z^2(N)) - 2E_Q(Z(N)) + 1 \\ &= E_Q(Z^2(N)) - 2 + 1 \\ &= E_Q(Z^2(N)) - 1 \\ &= \sum_{n=1}^N \sum_{1 \leq i_1 < \dots < i_n \leq N} c_{N,1}^{2n} \sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0^2(i_k - i_{k-1}, x_k - x_{k-1}) \\ &\leq \sum_{n=1}^N \left(c_1 c_{N,1}^2 N^{1/2} \right)^n \end{aligned}$$

where in the third equality we use $E_Q(Z(N)) = 1$ because the $h(n, x)$ have mean 0, in the fifth equality we use that the 0-th term in the second moment expansion of $Z(N)$ is 1 (see Lemma 2.1) and the last inequality follows from Proposition 1.2 i). \square

Lemma 3.10.

$$E_Q((K(N) - N)^2) \leq N^2 \sum_{n=1}^N \left(c_1 c_{N,1}^2 N^{1/2} \right)^n$$

Proof.

$$\begin{aligned} E_Q((K(N) - N)^2) &= E_Q(K^2(N) - 2K(N)N + N^2) \\ &= E_Q(K^2(N)) - 2NE_Q(K(N)) + N^2 \\ &= E_Q(K^2(N)) - 2N^2 + N^2 \\ &= E_Q(K^2(N)) - N^2 \\ &= \sum_{n=1}^N \sum_{1 \leq i_1 < \dots < i_n \leq N} c_{N,1}^{2n} \sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0^2(i_k - i_{k-1}, x_k - x_{k-1}) \\ &\quad \times \left(\sum_x x^2 p_0(N - i_n, x - x_n) \right)^2 \\ &\leq N^2 \sum_{n=1}^N \left(c_1 c_{N,1}^2 N^{1/2} \right)^n \end{aligned}$$

where in the third equality we use $E_Q(K(N)) = N$ because the $h(n, x)$ has mean zero and second moment of simple random walk of length N in dimension $d = 1$ is N , in the fifth

equality we use that the 0-th term in the second moment expansion of $K(N)$ is N^2 (see Lemma 2.2) and the last inequality follows from Proposition 1.2 ii). \square

Lemma 3.11. $Z(N) \rightarrow 1$ in probability as $N \rightarrow \infty$

Proof. For any $\epsilon > 0$, by Chebyshev's inequality and using Lemma 3.9, we have

$$P(|Z(N) - 1| > \epsilon) \leq \frac{E_Q((Z(N) - 1)^2)}{\epsilon^2} \leq \frac{\sum_{n=1}^N \left(c_1 c_{N,1}^2 N^{1/2}\right)^n}{\epsilon^2}$$

Let $f(N) = c_1 c_{N,1}^2 N^{1/2}$. By choice of $c_{N,1}$ (see (1.1)), $\lim_{N \rightarrow \infty} c_{N,1}^2 N^{1/2} = 0$, in particular $\lim_{N \rightarrow \infty} f(N) = 0$, which says given $\delta > 0$ small, there exists K such that for $N \geq K$, $f(N) < \delta$ (note that $f(N) \geq 0$), but then

$$S_N := \sum_{n=1}^N f(N)^n < \sum_{n=1}^N \delta^n = \frac{\delta - \delta^{N+1}}{1 - \delta} < \frac{\delta}{1 - \delta}$$

Since $\delta > 0$ is arbitrary and for large N , S_N is arbitrarily small, so $S_N \rightarrow 0$ as $N \rightarrow \infty$. We conclude $Z(N) \rightarrow 1$ in probability as $N \rightarrow \infty$. \square

Lemma 3.12. $\frac{1}{Z(N)} \rightarrow 1$ in probability as $N \rightarrow \infty$

Proof. By Lemma 3.11, for $\epsilon > 0$ small such that $1 - \epsilon > 0$ and given $\delta > 0$, there exists K such that for $N \geq K$, $P(|Z(N) - 1| \leq \epsilon) = 1 - P(|Z(N) - 1| > \epsilon) > 1 - \delta$. But

$$\begin{aligned} P(|Z(N) - 1| \leq \epsilon) &= P(1 - \epsilon \leq Z(N) \leq 1 + \epsilon) \\ &= P\left(\frac{1}{1 + \epsilon} \leq \frac{1}{Z(N)} \leq \frac{1}{1 - \epsilon}\right) \\ &= P\left(1 - \epsilon'' \leq \frac{1}{Z(N)} \leq 1 + \epsilon'\right) \\ &\leq P\left(1 - \hat{\epsilon} \leq \frac{1}{Z(N)} \leq 1 + \hat{\epsilon}\right) \end{aligned}$$

where first equality holds because we assume $\epsilon > 0$ is small such that $1 - \epsilon > 0$, and $\epsilon' = \frac{1}{1 - \epsilon} - 1 > 0$, $\epsilon'' = 1 - \frac{1}{1 + \epsilon} > 0$, $\hat{\epsilon} = \max\{\epsilon', \epsilon''\}$, so $1 - \delta \leq P\left(1 - \hat{\epsilon} \leq \frac{1}{Z(N)} \leq 1 + \hat{\epsilon}\right)$ and we have $P\left(\left|\frac{1}{Z(N)} - 1\right| > \epsilon\right) \rightarrow 0$ for all $\epsilon > 0$ small. But we note for $\epsilon' > \epsilon$, $P\left(\left|\frac{1}{Z(N)} - 1\right| > \epsilon'\right) \leq P\left(\left|\frac{1}{Z(N)} - 1\right| > \epsilon\right)$, so $P\left(\left|\frac{1}{Z(N)} - 1\right| > \epsilon\right) \rightarrow 0$ holds for any $\epsilon > 0$. Thus $\frac{1}{Z(N)} \rightarrow 1$ in probability as $N \rightarrow \infty$. \square

Lemma 3.13. $\frac{K(N)}{N} \rightarrow 1$ in probability as $N \rightarrow \infty$

Proof. For any $\epsilon > 0$, by Chebyshev inequality and using Lemma 3.10, we have

$$P\left(\left|\frac{K(N)}{N} - 1\right| > \epsilon\right) = P(|K(N) - N| > N\epsilon) \leq \frac{E_Q((K(N) - N)^2)}{\epsilon^2 N^2} \leq \frac{N^2 \sum_{n=1}^N \left(c_1 c_{N,1}^2 N^{1/2}\right)^n}{\epsilon^2 N^2}$$

As before since $\lim_{N \rightarrow \infty} \sum_{n=1}^N \left(c_1 c_{N,1}^2 N^{1/2}\right)^n = 0$, we see $\frac{K(N)}{N} \rightarrow 1$ in probability as $N \rightarrow \infty$. \square

Now we are ready to show Theorem 1.1 for dimension $d = 1$.

Since multiplication preserves convergence in probability, given $X_n \rightarrow X$ and $Y_n \rightarrow Y$ in probability, then $X_n \cdot Y_n \rightarrow X \cdot Y$ in probability, and recall that the mean square displacement of the polymer is $\langle \omega(N)^2 \rangle_{N,h} = \frac{K(N)}{Z(N)}$, then

$$\frac{\langle \omega(N)^2 \rangle_{N,h}}{N} = \frac{K(N)}{N} \cdot \frac{1}{Z(N)}$$

By Lemma 3.12, $\frac{1}{Z(N)} \rightarrow 1$ in probability and by Lemma 3.13, $\frac{K(N)}{N} \rightarrow 1$ in probability, we conclude $\frac{\langle \omega(N)^2 \rangle_{N,h}}{N} \rightarrow 1$ in probability.

4. DIFFUSIVITY OF RESCALED RANDOM POLYMER IN $d = 2$

In this section, we are going to show Proposition 1.3 and Theorem 1.1 for dimension $d = 2$.

Section 4.1. In this subsection, we are going to show Proposition 1.3 i) in a series of lemmas.

Lemma 4.1.

$$\sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0^2(i_k - i_{k-1}, x_k - x_{k-1}) \leq c_2^n \prod_{k=1}^n (i_k - i_{k-1})^{-1}$$

for some constant c_2 that depends only on the dimension $d = 2$

Proof. For $d = 2$, we see (3.2) is at most $c_2 n^{-1}$ for some constant c_2 . As in the proof of Lemma 3.1 using this uniform estimate for the $p_0(i_k - i_{k-1}, x_k - x_{k-1})$'s, we have

$$\begin{aligned} \sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0^2(i_k - i_{k-1}, x_k - x_{k-1}) &= \sum_{x_1} p_0^2(i_1, x_1) \cdots \sum_{x_n} p_0^2(i_n - i_{n-1}, x_n - x_{n-1}) \\ &\leq c_2^n i_1^{-1} \cdots (i_n - i_{n-1})^{-1} \sum_{x_1} p_0(i_1, x_1) \cdots \sum_{x_n} p_0(i_n - i_{n-1}, x_n - x_{n-1}) = c_2^n \prod_{k=1}^n (i_k - i_{k-1})^{-1} \end{aligned}$$

for some constant c_2 that depends only on the dimension $d = 2$ □

Lemma 4.2.

$$\sum_{1 \leq i_1 < \dots < i_n \leq N} c_2^n c_{N,2}^{2n} \prod_{k=1}^n (i_k - i_{k-1})^{-1} \leq (c_2 c_{N,2}^2 \log N)^n$$

Proof.

$$\begin{aligned} &\sum_{1 \leq i_1 < \dots < i_n \leq N} c_2^n c_{N,2}^{2n} \prod_{k=1}^n (i_k - i_{k-1})^{-1} \\ &= c_2^n c_{N,2}^{2n} \sum_{i_1=1}^{N-(n-1)} \cdots \sum_{i_{n-1}=i_{n-2}+1}^{N-1} i_1^{-1} \cdots (i_{n-1} - i_{n-2})^{-1} \sum_{i_n=i_{n-1}+1}^N (i_n - i_{n-1})^{-1} \end{aligned}$$

$$\leq c_2^n c_{N,2}^{2n} \sum_{i_1=1}^{N-(n-1)} \cdots \sum_{i_{n-1}=i_{n-2}+1}^{N-1} i_1^{-1} \cdots (i_{n-1} - i_{n-2})^{-1} 10 \log N$$

Last inequality holds because $\sum_{k=1}^N k^{-1} \leq 1 + \int_1^N x^{-1} dx = 1 + \log N \leq 10 \log N$. Continuing from above and arguing similarly to estimate each sum in the expression we have

$$\leq c_2^n c_{N,2}^{2n} \log N \sum_{i_1=1}^{N-(n-1)} \cdots \sum_{i_{n-1}=i_{n-2}+1}^{N-1} i_1^{-1} \cdots (i_{n-1} - i_{n-2})^{-1} \leq (c_2 c_{N,2}^2 \log N)^n$$

where the constant c_2 will change from line to line (again it depends only on the dimension $d = 2$). \square

We conclude by Lemma 2.1 that Proposition 1.3 i) holds.

Section 4.2. In this subsection, we are going to show Proposition 1.3 ii) in a series of lemmas.

By standard computations of the partial moments of simple random walk of length n in dimension $d = 2$ using characteristic function, for $x = (x_1, x_2)$, we have

$$\sum_x x_1^2 p_0(n, x) = \frac{n}{2}; \quad \sum_x x_1^4 p_0(n, x) = \frac{3n^2 - n}{4}; \quad \sum_x x_1^2 x_2^2 p_0(n, x) = \frac{n(n-1)}{4}$$

so the second and fourth moments are respectively

$$\sum_x |x|^2 p_0(n, x) = n; \quad \sum_x |x|^4 p_0(n, x) = 2n^2 - n$$

Lemma 4.3.

$$\sum_x |x|^2 p_0(N - i_n, x - x_n) = (N - i_n) + |x_n|^2$$

Proof.

$$\begin{aligned} \sum_x |x|^2 p_0(N - i_n, x - x_n) &= \sum_{x-x_n} |x|^2 p_0(N - i_n, x - x_n) = \sum_x |x + x_n|^2 p_0(N - i_n, x) \\ &= \sum_x |x|^2 p_0(N - i_n, x) + \sum_x |x_n|^2 p_0(N - i_n, x) = (N - i_n) + |x_n|^2 \end{aligned}$$

where third equality holds because any odd partial moments of simple random walk vanish. \square

Lemma 4.4.

$$\sum_{x_k} |x_k|^4 p_0(i_k - i_{k-1}, x_k - x_{k-1}) = 2(i_k - i_{k-1})^2 - (i_k - i_{k-1}) + |x_{k-1}|^4 + 4|x_{k-1}|^2(i_k - i_{k-1})$$

Proof.

$$\begin{aligned} &\sum_{x_k} |x_k|^4 p_0(i_k - i_{k-1}, x_k - x_{k-1}) \\ &= \sum_{x_k - x_{k-1}} |x_k|^4 p_0(i_k - i_{k-1}, x_k - x_{k-1}) = \sum_{x_k} |x_k + x_{k-1}|^4 p_0(i_k - i_{k-1}, x_k) \end{aligned}$$

$$\begin{aligned}
&= \sum_{x_k} |x_k|^4 p_0(i_k - i_{k-1}, x_k) + |x_{k-1}|^4 \sum_{x_k} p_0(i_k - i_{k-1}, x_k) + 2|x_{k-1}|^2 \sum_{x_k} |x_k|^2 p_0(i_k - i_{k-1}, x_k) \\
&\quad + 4x_{k-1,1}^2 \sum_{x_k} x_{k,1}^2 p_0(i_k - i_{k-1}, x_k) + 4x_{k-1,2}^2 \sum_{x_k} x_{k,2}^2 p_0(i_k - i_{k-1}, x_k) \\
&= 2(i_k - i_{k-1})^2 - (i_k - i_{k-1}) + |x_{k-1}|^4 + 2|x_{k-1}|^2(i_k - i_{k-1}) + 4x_{k-1,1}^2 \frac{i_k - i_{k-1}}{2} + 4x_{k-1,2}^2 \frac{i_k - i_{k-1}}{2} \\
&= 2(i_k - i_{k-1})^2 - (i_k - i_{k-1}) + |x_{k-1}|^4 + 4|x_{k-1}|^2(i_k - i_{k-1})
\end{aligned}$$

□

Lemma 4.5.

$$\begin{aligned}
&\sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0(i_k - i_{k-1}, x_k - x_{k-1}) ((N - i_n)^2 + 2(N - i_n)|x_n|^2 + |x_n|^4) \\
&= (N - i_n)^2 + 2(N - i_n)i_n + 2 \sum_{k=1}^n (i_k - i_{k-1})^2 - i_n + 4 \sum_{k=1}^n (i_k - i_{k-1})i_{k-1}
\end{aligned}$$

Proof. It follows by induction on n as in Lemma 3.5

□

Lemma 4.6.

$$\sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0^2(i_k - i_{k-1}, x_k - x_{k-1}) \left(\sum_x |x|^2 p_0(N - i_n, x - x_n) \right)^2 \leq c_2^n N^2 \prod_{k=1}^n (i_k - i_{k-1})^{-1}$$

Proof. Using the uniform estimate (3.2) for $d = 2$ on the transition probability as in the proof of Lemma 4.1 we get

$$\begin{aligned}
&\sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0^2(i_k - i_{k-1}, x_k - x_{k-1}) \left(\sum_x |x|^2 p_0(N - i_n, x - x_n) \right)^2 \\
&\leq c_2^n i_1^{-1} (i_2 - i_1)^{-1} \cdots (i_n - i_{n-1})^{-1} \sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0(i_k - i_{k-1}, x_k - x_{k-1}) ((N - i_n) + |x_n|^2)^2 \\
&= c_2^n i_1^{-1} (i_2 - i_1)^{-1} \cdots (i_n - i_{n-1})^{-1} \sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0(i_k - i_{k-1}, x_k - x_{k-1}) \\
&\quad \times ((N - i_n)^2 + 2(N - i_n)|x_n|^2 + |x_n|^4) \\
&= c_2^n i_1^{-1} (i_2 - i_1)^{-1} \cdots (i_n - i_{n-1})^{-1} \\
&\quad \times \left((N - i_n)^2 + 2(N - i_n)i_n + 2 \sum_{k=1}^n (i_k - i_{k-1})^2 - i_n + 4 \sum_{k=1}^n (i_k - i_{k-1})i_{k-1} \right) \\
&\leq c_2^n N^2 \prod_{k=1}^n (i_k - i_{k-1})^{-1}
\end{aligned}$$

where in the first inequality we also use Lemma 4.3 to compute the second moment, in the second equality we use Lemma 4.5 and in the last inequality we use estimate similar to that in Lemma 3.6. □

Lemma 4.7.

$$\sum_{1 \leq i_1 < \dots < i_n \leq N} c_2^n c_{N,2}^{2n} N^2 \prod_{k=1}^n (i_k - i_{k-1})^{-1} \leq N^2 (c_2 c_{N,2}^2 \log N)^n$$

Proof. It follows from Lemma 4.2. □

We conclude by Lemma 2.2 that Proposition 1.3 ii) holds.

Section 4.3. In this subsection, we are going to show Theorem 1.1 for $d = 2$.

Clearly, Lemmas 3.9 and 3.10 hold for dimension $d = 2$ with $c_2 c_{N,2}^2 \log N$ instead of $c_1 c_{N,1}^2 N^{1/2}$, precisely we have

$$E_Q((Z(N) - 1)^2) \leq \sum_{n=1}^N (c_2 c_{N,2}^2 \log N)^n; \quad E_Q((K(N) - N)^2) \leq N^2 \sum_{n=1}^N (c_2 c_{N,2}^2 \log N)^n$$

By choice of $c_{N,2}$ (see (1.1)), as in the proof of Lemma 3.11 it implies

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N (c_2 c_{N,2}^2 \log N)^n = 0$$

thus we see Lemmas 3.11, 3.12 and 3.13 hold with suitable changes. We conclude that for dimension $d = 2$, $\frac{\langle \omega(N)^2 \rangle_{N,h}}{N} \rightarrow 1$ in probability.

5. OTHER RESULTS

In this section, we are going to show other results in the diffusive regime.

Theorem 5.1. *With rescaling of the polymer density by $c_{N,d}$ for $d = 1, 2$, there exists normalizing constants $a_{N,d}$ such that*

$$a_{N,d} (Z_N - 1) \Rightarrow \xi$$

where ξ is some Gaussian random variable.

First, we have the following lemma.

Lemma 5.2. *For $x_1, \dots, x_n \in \mathbb{Z}^d$,*

$$\sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0^2(i_k - i_{k-1}, x_k - x_{k-1}) = \prod_{k=1}^n p_0(2(i_k - i_{k-1}), 0)$$

Proof. Note that for transition probability $p_0(n, x)$ of the simple random walk starting at 0 and ending at x at time n , by spatial homogeneity it is same as the transition probability $p_x(n, 2x)$ of the simple random walk starting at x and ending at $2x$ at time n . Furthermore, by reflecting each step walk takes to reach from x to $2x$, for example in dimension $d = 2$ if original walk goes up, then the reflecting walk goes down and if original walk goes right,

then the reflecting walk goes left, we get a reflecting walk starting at x and ending at 0 at time n with transition probability $p_x(n, 0)$ such that

$$p_0(n, x) = p_x(n, 2x) = p_x(n, 0)$$

Using Chapman-Kolmogorov equality for the simple random walk, we have

$$\sum_x p_0^2(n, x) = \sum_x p_0(n, x) p_0(n, x) = \sum_x p_0(n, x) p_x(n, 0) = p_0(2n, 0)$$

We conclude that

$$\begin{aligned} & \sum_{x_1, \dots, x_n} \prod_{k=1}^n p_0^2(i_k - i_{k-1}, x_k - x_{k-1}) \\ &= \sum_{x_1} p_0^2(i_1, x_1) \cdots \sum_{x_{n-1}} p_0^2(i_{n-1} - i_{n-2}, x_{n-1} - x_{n-2}) \sum_{x_n} p_0^2(i_n - i_{n-1}, x_n - x_{n-1}) \\ &= \sum_{x_1} p_0^2(i_1, x_1) \cdots \sum_{x_{n-1}} p_0^2(i_{n-1} - i_{n-2}, x_{n-1} - x_{n-2}) \sum_{x_n - x_{n-1}} p_0^2(i_n - i_{n-1}, x_n - x_{n-1}) \\ &= \sum_{x_1} p_0^2(i_1, x_1) \cdots \sum_{x_{n-1}} p_0^2(i_{n-1} - i_{n-2}, x_{n-1} - x_{n-2}) \sum_{x_n} p_0^2(i_n - i_{n-1}, x_n) \\ &= \sum_{x_1} p_0^2(i_1, x_1) \cdots \sum_{x_{n-1}} p_0^2(i_{n-1} - i_{n-2}, x_{n-1} - x_{n-2}) p_0(2(i_n - i_{n-1}), 0) \\ &= \prod_{k=1}^n p_0(2(i_k - i_{k-1}), 0) \end{aligned}$$

□

To show Theorem 5.1, we write the partition function as a sum of two parts

Lemma 5.3. $Z_N - 1 = \sum_{k=1}^N f_k + R_N$

Proof.

$$\begin{aligned} Z_N - 1 &= g_1 + \sum_{n=2}^N g_n \\ &= \int \sum_{k=1}^N c_{N,d} h(k, \omega(k)) dP_0^N(\omega) + \sum_{n=2}^N g_n \\ &= \sum_{k=1}^N \sum_x c_{N,d} h(k, x) p_0(k, x) + R_N \\ &= \sum_{k=1}^N f_k + R_N \end{aligned}$$

where the g_n are as in Lemma 2.1

□

By the following two propositions, Theorem 5.1 follows since if $X_n \Rightarrow X$, $Y_n \Rightarrow a$ where a is a constant, then $X_n + Y_n \Rightarrow X + a$.

Proposition 5.4. $\sum_{k=1}^N a_{N,d} f_k \Rightarrow \xi$ where ξ is some Gaussian random variable.

Proof. By definition, $f_k = c_{N,d} \sum_x h(k, x) p_0(k, x)$, if we let $X_{k,N} = a_{N,d} f_k$, then to show proposition it suffices to check conditions in the Lindeberg-Feller Theorem are satisfied, i.e. $\sum_{k=1}^N E_Q X_{k,N}^2 \rightarrow c$ where $c > 0$, and for all $\epsilon > 0$, $\lim_N \sum_{k=1}^N E_Q \left(X_{k,N}^2; |X_{k,N}| > \epsilon \right) = 0$. By direct computations, we find

$$\begin{aligned} E_Q X_{k,N}^2 &= a_{N,d}^2 c_{N,d}^2 E_Q \left(\sum_x h^2(k, x) p_0^2(k, x) + \sum_{x \neq x'} h(k, x) h(k, x') p_0(k, x) p_0(k, x') \right) \\ &= a_{N,d}^2 c_{N,d}^2 \sum_x p_0^2(k, x) \\ &= a_{N,d}^2 c_{N,d}^2 p_0(2k, 0) \end{aligned}$$

where last equality follows from Lemma 5.2. Using estimate of the transition probability $p_0(n, x)$ as in 3.2, in $d = 1$, $p_0(2k, 0) = \pi^{-1/2} k^{-1/2} + r_{2k}(0)$, where $|r_{2k}(0)| \leq c_1 k^{-3/2}$ and in $d = 2$, $p_0(2k, 0) = \pi^{-1} k^{-1} + r_{2k}(0)$, where $|r_{2k}(0)| \leq c_2 k^{-2}$, we see in $d = 2$, $\sum_{k=1}^N E_Q X_{k,N}^2 = a_{N,2}^2 c_{N,2}^2 \sum_{k=1}^N p_0(2k, 0) = a_{N,2}^2 c_{N,2}^2 \left(\sum_{k=1}^N \pi^{-1} k^{-1} + r_{2k}(0) \right)$. If we take $a_{N,2} = \left(c_{N,2}^2 \log N \right)^{-1/2}$, then

$$a_{N,2}^2 c_{N,2}^2 \left(\sum_{k=1}^N \pi^{-1} k^{-1} + r_{2k}(0) \right) \rightarrow \pi^{-1}$$

(because $1 - (N+1)^{-1} \leq \sum_{k=1}^N k^{-2} \leq 2 - N^{-1}$, so $a_{N,2}^2 c_{N,2}^2 \sum_{k=1}^N k^{-2} \rightarrow 0$).

Next, for given $\epsilon > 0$, we also find

$$\begin{aligned} E_Q \left(X_{k,N}^2; |X_{k,N}| \geq \epsilon \right) &= E_Q \left(a_{N,2}^2 f_k^2; a_{N,2} |f_k| > \epsilon \right) \\ &\leq a_{N,2}^2 c_{N,2}^2 E_Q \left(\sum_x p_0^2(k, x) 1_{a_{N,2} |f_k| > \epsilon} \right) \\ &\quad + a_{N,2}^2 c_{N,2}^2 E_Q \left(\sum_{x \neq x'} |h(k, x) h(k, x')| p_0(k, x) p_0(k, x') 1_{a_{N,2} |f_k| > \epsilon} \right) \\ &= a_{N,2}^2 c_{N,2}^2 Q(a_{N,2} |f_k| > \epsilon) \left(p_0(2k, 0) + \sum_{x \neq x'} p_0(k, x) p_0(k, x') \right) \end{aligned}$$

But $Q(a_{N,2} |f_k| > \epsilon) = Q(|\sum_x h(k, x) p_0(k, x)| > a_{N,2}^{-1} c_{N,2}^{-1} \epsilon) \leq Q(\sum_x |h(k, x)| p_0(k, x) > a_{N,2}^{-1} c_{N,2}^{-1} \epsilon) = Q(1 > a_{N,2}^{-1} c_{N,2}^{-1} \epsilon) = Q(1 > (\log N)^{1/2} \epsilon) = 0$ for N large. Thus

$$\lim_N \sum_{k=1}^N E_Q \left(X_{k,N}^2; |X_{k,N}| > \epsilon \right) = 0$$

Similarly, we can check conditions in the Lindeberg-Feller theorem are satisfied for $d = 1$ if we take $a_{N,1} = (c_{N,1}^2 N^{1/2})^{-1/2}$. \square

Proposition 5.5. $a_{N,d} R_N \Rightarrow 0$

Proof. Note that $a_{N,d} R_N \Rightarrow 0$ if and only if $a_{N,d} R_N \rightarrow 0$ in probability and $a_{N,d} R_N \rightarrow 0$ in probability if $E_Q (a_{N,d} R_N)^2 \rightarrow 0$ by Chebyshev inequality.

We show the proposition for $d = 2$, and it is similar for $d = 1$.

By definition, $R_N = \sum_{n=2}^N g_n$, so $E_Q(a_{N,d}R_N)^2 = a_{N,d}^2 E_Q\left(\sum_{n=2}^N g_n^2 + \sum_{n \neq m} g_n g_m\right)$. From Lemma 2.1, we know that for $n \neq m$, $E_Q g_n g_m = 0$, and $E_Q g_n^2 = \sum_{1 \leq i_1 < \dots < i_n \leq N} c_{N,2}^{2n} \prod_{k=1}^n p_0(2(i_k - i_{k-1}), 0) \leq \left(c_2 c_{N,2}^2 \log N\right)^n$. Recall $a_{N,2} = \left(c_{N,2}^2 \log N\right)^{-1/2}$ for $d = 2$ so

$$a_{N,2}^2 E_Q \sum_{n=2}^N g_n^2 \leq \sum_{n=2}^N a_{N,2}^2 (c_2 c_{N,2}^2 \log N)^n = \sum_{n=2}^N c_2 (c_2 c_{N,2}^2 \log N)^{n-1} \rightarrow 0$$

□

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